

Price Inequalities and Betti Number Growth on Manifolds without Conjugate Points

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Abstract

We derive Price inequalities for harmonic forms on manifolds without conjugate points and with a negative Ricci upper bound. The techniques employed in the proof work particularly well for manifolds of non-positive sectional curvature, and in this case we prove a strengthened Price inequality. We employ these inequalities to study the asymptotic behavior of the Betti numbers of coverings of Riemannian manifolds without conjugate points. Finally, we give a vanishing result for L^2 -Betti numbers of closed manifolds without conjugate points.

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1 Introduction

In the early 1980's, Donnelly and Xavier introduced an integral inequality [DX84, Theorem 2.2] which became a standard tool in the proof of cohomology vanishing theorems and spectral bounds on simply connected negatively curved manifolds. Around the same time Price introduced an inequality [Pri83] which became ubiquitous in the study of singularities of harmonic maps and Yang-Mills fields. Price's inequality can be proved in a manner which generalizes the proof of Donnelly and Xavier's inequality. In this work, we further this circle of ideas by applying Price's generalization of the Donnelly-Xavier inequality to study cohomology, obtaining new bounds on the Betti numbers of Riemannian manifolds with negative Ricci curvature and no conjugate points. Our approach is quite robust and works without assuming pinched negative curvature. In particular, the hypotheses of the main technical estimate, Theorem 38, allow some positive sectional curvature. Of course, when we do impose stronger pinched negative sectional curvature assumptions, we are able to derive a strengthened Price inequality, Theorem 84.

We apply these Price inequalities to study the Betti numbers of closed Riemannian manifolds without conjugate points. In particular, we obtain asymptotic bounds on the growth of certain Betti numbers in towers of regular coverings of such manifolds. The range of Betti numbers to which these result apply depends on how negative the Ricci curvature is assumed to be.

The study of the asymptotic behavior of Betti numbers has attracted considerable interest in the last four decades, especially in the context of coverings of locally symmetric spaces of non-compact type. See, for example, [DW78],

[DW79], [Xue91], [SX91], [ABBGNRS12], [Mar14]. Many of the algebraic techniques employed in these works are not obviously amenable to generalization outside the locally symmetric context.

Given a co-compact torsion free lattice Γ acting on a symmetric space of non-compact type say G/K , we say a sequence of nested, normal, finite index subgroups $\{\Gamma_i\}$ of Γ is a *cofinal filtration* of Γ if $\bigcap_i \Gamma_i$ is the identity element. Any such Γ is known to be *residually finite* (see [Bor63, Proposition 2.3]), so that cofinal filtrations always exists. Denote by M_i the finite index regular cover of $\Gamma \backslash G/K$ associated to Γ_i . It follows from the results of [DW78], [DW79], that for any cofinal filtration of Γ , for $k \neq \frac{1}{2} \dim(G/K)$,

$$\lim_{i \rightarrow \infty} \frac{b_k(M_i)}{\text{Vol}(M_i)} = 0, \quad (1)$$

where $b_k(M_i)$ denotes the k -th Betti number of M_i .

Our techniques provide new methods for proving such results and quantifying the sub volume order growth of certain Betti numbers. Because our techniques do not rely on representation theory or the trace formula, it is natural to consider these questions on manifolds which are *not* locally symmetric. The Price inequalities given in Theorem 38 and Proposition 64 are tailored to address such problems. For example, we have the following *non-locally symmetric* analog of the DeGeorge-Wallach result (1).

Theorem 2 (See Corollary 75). *Let (M^n, g) be a closed Riemannian manifold without conjugate points with $-1 \leq \sec_g \leq 1$. Assume there exists $\delta > 4k^2$ such that*

$$-Ric \geq \delta g.$$

Let $\pi_i : M_i \rightarrow M$ be a sequence of Riemannian covers of M with injectivity radii, denoted $\gamma_{g_i}(M_i)$, satisfying $\gamma_{g_i}(M_i) \rightarrow \infty$. Then there exists $b(n, k, \delta) > 0$ so that for $\gamma_{g_i}(M_i)$ sufficiently large,

$$\frac{b_k(M_i)}{\text{Vol}(M_i)} \leq b(n, k, \delta) e^{-(\sqrt{\delta}-2k)\gamma_{g_i}(M_i)}. \quad (3)$$

In particular,

$$\lim_{i \rightarrow \infty} \frac{b_k(M_i)}{\text{Vol}(M_i)} = 0. \quad (4)$$

Observe that the range of Betti numbers covered by Theorem 2 grows as the square root of the the lower bound δ . Under our curvature normalization, δ can at most be $n - 1$ with equality if and only if the underlying Riemannian manifold is real hyperbolic with sectional curvature -1 . Thus, Theorem 2 does not address the full range of Betti numbers achieved by DeGeorge and Wallach in the locally symmetric case. On the other hand, Theorem 2 is not only free of any homogeneity requirement on the metric, but also does not require any direct assumption on the sectional curvature.

Given a closed manifold (M, g) without conjugate points and infinite residually finite fundamental group, the injectivity radius goes to infinity in any tower of regular Riemannian covers associated to a cofinal filtration of its fundamental group. (See [DW78, Theorem 2.1].) Hence Theorem 2 not only implies (1) for certain k -th Betti numbers, but also implies $\frac{b_k(M_i)}{\text{Vol}_g(M_i)}$ decays exponentially in the injectivity radii $\gamma_{g_i}(M_i)$. Such decay results have been obtained by numerous authors in the locally symmetric context. See for example [Xue91], [SX91], [Mar14]. Outside the locally symmetric space context, related results have also been obtained by [BLLS14] for p -adically defined towers, for which they show that if $\lim_{i \rightarrow \infty} \frac{b_k(M_i)}{\text{Vol}_g(M_i)} = 0$, then $\frac{b_k(M_i)}{\text{Vol}_g(M_i)}$ decays like $\text{Vol}(M_i)^{-d}$, for a specific dimensional constant d . In a similar vein, we can reexpress the decay results of Theorem 2 in terms of volumes under suitable hypotheses on the injectivity radii of Riemannian coverings associated to a cofinal filtration. We refer to such an assumption on the cofinal filtration as “congruence type”, see Definition 78. Under such assumptions Theorem 2 immediately yields the following corollary.

Corollary 5. *Let (M^n, g) be a Riemannian manifold without conjugate points with $-1 \leq \sec_g \leq 1$ and residually finite fundamental group Γ . Assume*

$$-\text{Ric} \geq \delta g, \quad \delta > 4k^2.$$

For any congruence type cofinal filtration $\{\Gamma_i\}$ of Γ of exponent α , if we denote by $\pi_i : M_i \rightarrow M$ the regular Riemannian cover of M associated to Γ_i , we have for $\gamma_g(M)$, sufficiently large

$$b_k(M_i) \leq d(n, k, \delta, \Gamma) \text{Vol}(M_i)^{1-2\alpha(\frac{\sqrt{\delta}}{2}-k)}.$$

where $d(n, k, \Gamma)$ is a positive constant.

If one assumes the sectional curvature to be negative and suitably pinched, Theorem 5 can be considerably strengthened. See, for example, Corollary 103. We obtain our sharpest results for real hyperbolic space. See Theorem 111. In a subsequent paper, we will show how to modify our techniques to obtain stronger estimates for Betti numbers of the complex and quaternionic hyperbolic spaces than follow from our results in this paper.

Finally, our techniques can also be applied to L^2 -cohomology problems. For example, we have the following vanishing result for L^2 -Betti numbers of closed manifolds without conjugate points.

Theorem 6 (See Theorem 117). *Let (M^n, g) be a closed Riemannian manifold without conjugate points and $-1 \leq \sec_g \leq 1$. If there exists $\delta > 4k^2$ such that*

$$-\text{Ric} \geq \delta g,$$

then the k -th L^2 -Betti number of M vanishes.

Theorem 6 provides further evidence towards the Singer Conjecture for aspherical manifolds. For the details of the proof and more on the Singer Conjecture we refer to Section 8.

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2 Some Integral Equalities for Harmonic Forms

In this section, we fix notation and derive the Price/Donnelly-Xavier inequality for harmonic k -forms. We also collect a few facts regarding Moser iteration which will be used throughout this work.

Let (M^n, g) be a complete n -dimensional Riemannian manifold, with injectivity radius $\gamma_g(M)$. Given a point $p \in M$, denote by $B_R(p)$ the geodesic ball of radius $0 < R \leq \gamma_g(M)$ centered at p . In such a ball, let $g = dr^2 + g_r$ be the expression of the metric g in geodesic polar coordinates, and let ∂_r denote the unit radial vector field.

Given a 1-form ϕ , let $e(\phi)$ denote exterior multiplication on the left by ϕ . Let $e^*(\phi)$ denote the adjoint operator. Fix a local orthonormal frame $\{e_j\}_j$ and coframe $\{\omega^j\}_j$. Acting on forms of arbitrary degree, the Lie derivative in the radial direction can be written as

$$L_{\partial_r} = \{d, e^*(dr)\} = \nabla_{\partial_r} + e(\omega^j)e^*(\nabla_{e_j}dr). \quad (7)$$

Choosing the orthonormal frame so that $e_n = \partial_r$, we may express $\nabla_{e_j}dr$ in terms of the second fundamental form h of $S_r(p)$. Recall for X and Y tangent to S_r ,

$$h(X, Y) := g(\nabla_X \partial_r, Y).$$

Thus, we write

$$\nabla_{e_i}dr = h_{i1}\omega^1 + \dots + h_{in-1}\omega^{n-1},$$

and

$$e(\omega^j)e^*(\nabla_{e_j}dr) = \sum_{j,k < n} h_{jk}e(\omega^j)e^*(\omega^k) =: Q. \quad (8)$$

The operator Q defined by (8) is the natural extension of the second fundamental form to an endomorphism of forms of arbitrary degree. As usual, we set the mean curvature of the geodesic sphere S_r to be the trace $H = \sum_k h_{kk}$.

Let $\mathcal{H}_g^k(M)$ denote the strongly harmonic k -forms on M . Given $\alpha \in \mathcal{H}_g^k(M)$,

we have

$$\begin{aligned}
\int_{B_R(p)} \langle L_{\partial_r} \alpha, \alpha \rangle dv &= \int_{B_R(p)} \langle d(e^*(dr)\alpha), \alpha \rangle dv \\
&= \int_{B_R(p)} \langle e^*(dr)\alpha, d^*\alpha \rangle dv + \int_{S_R(p)} |e^*(dr)\alpha|^2 d\sigma \\
&= \int_{S_R(p)} |e^*(dr)\alpha|^2 d\sigma,
\end{aligned} \tag{9}$$

where $d\sigma$ is the volume element on the geodesic sphere $S_R(p)$. Using (7) for L_{∂_r} gives the alternate expression

$$\begin{aligned}
\int_{B_R(p)} \langle L_{\partial_r} \alpha, \alpha \rangle dv &= \int_{B_R(p)} \langle (\nabla_{\partial_r} + Q)\alpha, \alpha \rangle dv \\
&= \int_{B_R(p)} \frac{1}{2} (L_{\partial_r} - Q)(|\alpha|^2) dv + \int_{B_R} \langle Q\alpha, \alpha \rangle dv \\
&= \int_{B_R(p)} \langle (Q - \frac{H}{2})\alpha, \alpha \rangle dv + \int_{S_R(p)} \frac{1}{2} |\alpha|^2 d\sigma.
\end{aligned} \tag{10}$$

Next, let us define the functions

$$\mu(r) := \frac{\int_{S_r(p)} |e^*(dr)\alpha|^2 d\sigma}{\int_{S_r(p)} |\alpha|^2 d\sigma}, \tag{11}$$

and

$$q(r) := \frac{\int_{S_r(p)} \langle (\frac{H}{2} - Q)\alpha, \alpha \rangle d\sigma}{\int_{S_r(p)} |\alpha|^2 d\sigma}. \tag{12}$$

Equating the expressions in (9) and (10) yields

$$\int_{B_R(p)} q(r) |\alpha|^2 dv = \frac{1}{2} \int_{S_R(p)} (1 - 2\mu(R)) |\alpha|^2 d\sigma. \tag{13}$$

Now if we multiply (13) by $\phi'(R)$, ϕ to be determined, and integrate from σ to $\tau \leq \gamma_g(M)$ we get

$$\begin{aligned}
&\phi(\tau) \int_{B_\tau(p)} q(r) |\alpha|^2 dv - \phi(\sigma) \int_{B_\sigma(p)} q(r) |\alpha|^2 dv \\
&= \int_{B_\tau(p) \setminus B_\sigma(p)} [\phi(r)q(r) + \frac{1}{2}\phi'(r)(1 - 2\mu(r))] |\alpha|^2 dv.
\end{aligned} \tag{14}$$

Next, we choose

$$\phi(r) := e^{-\int_\sigma^r \frac{q(s)ds}{\frac{1}{2} - \mu(s)}}. \tag{15}$$

in order to eliminate the last line of (14). Let us summarize this discussion into a proposition.

Proposition 16. *Let (M, g) be a Riemannian manifold. For any strongly harmonic k -form $\alpha \in \mathcal{H}_g^k(M)$ and for any $\sigma < \tau \leq \gamma_g(M)$, we have the Price equality*

$$\phi(\sigma) \int_{B_\sigma(p)} q(r) |\alpha|^2 dv = \phi(\tau) \int_{B_\tau(p)} q(r) |\alpha|^2 dv, \quad (17)$$

where $\mu(r)$, $q(r)$ and $\phi(r)$ are respectively defined as in (11), (12) and (15).

We conclude this section by studying the behavior of the function $\mu(r)$ for r close to zero.

Lemma 18. *Let (M, g) be a Riemannian manifold and $p \in M$. Given a k -form α such that $\alpha(p) \neq 0$, we have*

$$\lim_{r \rightarrow 0} \mu(r) = \frac{k}{n}. \quad (19)$$

Proof. This lemma does not require α to be harmonic. Fix geodesic coordinates with p at the origin. Write

$$\begin{aligned} |e^*(dr)\alpha|^2(x) &= \sum_{i,j} \frac{x^i x^j}{r^2} \langle e^*(dx^i)\alpha(x), e^*(dx^j)\alpha(x) \rangle \\ &= \sum_{i,j} \frac{x^i x^j}{r^2} \langle e^*(dx^i)\alpha(0), e^*(dx^j)\alpha(0) \rangle + o(1). \end{aligned}$$

Using

$$\int_{S_r} \frac{x^i x^j}{r^2} d\sigma = \frac{\delta_{ij}}{n} \text{Vol}(S_r),$$

and

$$\sum_j |e^*(dx^j)\alpha|^2(0) = k|\alpha|^2(0),$$

we see

$$\mu(r) = \frac{k}{n} + o(1).$$

□

In order to extract geometric information out of Proposition 16, we need to understand the positivity properties of $q(r)$. This is a problem in comparison geometry which we address in the next section.

3 Controlling the Second Fundamental Form

Let $\lambda_1 \geq \dots \geq \lambda_{n-1}$ denote the eigenvalues of h_{ij} . Q commutes with the decomposition of a k -form α as $\alpha = e^*(dr)e(dr)\alpha + e(dr)e^*(dr)\alpha$. Hence

$$\begin{aligned} \langle (\frac{H}{2} - Q)\alpha, \alpha \rangle &\geq \frac{1}{2}(-\lambda_1 - \dots - \lambda_k + \lambda_{k+1} + \dots + \lambda_{n-1})|e(dr)\alpha|^2 \\ &\quad + \frac{1}{2}(-\lambda_1 - \dots - \lambda_{k-1} + \lambda_k + \dots + \lambda_{n-1})|e^*(dr)\alpha|^2. \end{aligned} \quad (20)$$

With this notation,

$$\int_{S_R(p)} q(r)|\alpha|^2 d\sigma \geq \int_{S_R(p)} ((\frac{H}{2} - \sum_{i=1}^k \lambda_i)|\alpha|^2 + \mu \lambda_k |\alpha|^2) d\sigma. \quad (21)$$

In order to extract information from this inequality, we use the Rauch comparison theorem (see eg. [Pet15, p. 255]) and the Riccati Equation for the mean curvature of a geodesic sphere (see eg. [Pet15, Chapter 5]) to control the second fundamental form terms arising in the main inequality of Proposition 21. We first recall those results.

Let $\gamma(K)$ denote the injectivity radius of the space form of constant curvature K . Let

$$s_K(r) := \begin{cases} \frac{1}{\sqrt{K}} \sin(\sqrt{K}r) & \text{if } K > 0, \\ r & \text{if } K = 0, \\ \frac{1}{\sqrt{-K}} \sinh(\sqrt{-K}r) & \text{if } K < 0. \end{cases}$$

Theorem 22 (Rauch Comparison). *If (M^n, g) satisfies $K_1 \leq \sec_g \leq K_2$, $h(r)$ is the second fundamental form of the geodesic sphere of radius r , and $g = dr^2 + g_r$ denotes the metric in geodesic polar coordinates, then*

$$\frac{s'_{K_2}(r_1)}{s_{K_2}(r_1)} g_{r_1} \leq h(r_1), \text{ and } h(r_2) \leq \frac{s'_{K_1}(r_2)}{s_{K_1}(r_2)} g_{r_2},$$

for $0 < r_1 \leq \min\{\gamma_g(M), \gamma(K_2)\}$, and $0 < r_2 \leq \min\{\gamma_g(M), \gamma(K_1)\}$.

If we assume

$$-1 \leq \sec_g \leq \kappa, \quad (23)$$

Rauch's comparison becomes:

$$\cot(\kappa r_1) g_{r_1} \leq h(r_1), \text{ and } h(r_2) \leq \coth(r_2) g_{r_2}, \quad (24)$$

for $0 < r_1 \leq \min\{\gamma_g(M), \frac{\pi}{2\kappa}\}$, and $0 < r_2 \leq \gamma_g(M)$, which then implies the following bounds on the mean curvature $H(r, \sigma)$ of any geodesic sphere S_r .

$$(n-1) \cot(\kappa r_1) \leq H(r_1, \sigma), \text{ and } H(r_2, \sigma) \leq (n-1) \coth(r_2). \quad (25)$$

The Riccati equation for the mean curvature H is :

$$\partial_r H = -Ric(\partial_r, \partial_r) - |h|^2. \quad (26)$$

Since $H^2 \geq |h|^2$, we have

$$\partial_r H \geq -Ric(\partial_r, \partial_r) - H^2. \quad (27)$$

If we assume the Ricci curvature is negative and bounded away from zero, say

$$-Ric \geq \delta g, \quad (n-1) \geq \delta > 0, \quad (28)$$

then (27) implies

$$\partial_r H + H^2 \geq \delta. \quad (29)$$

(The upper bound $(n-1)$ on δ follows from the normalization (23) of the sectional curvature.) Next consider the ordinary differential equation saturating the inequality given in equation (29)

$$u' + u^2 = \delta.$$

One solution to this equation is given by

$$u(r) := \sqrt{\delta} \coth(\sqrt{\delta} r). \quad (30)$$

We now use a Riccati type comparison argument in conjunction with Rauch's comparison.

Lemma 31. *Let (M^n, g) be a closed Riemannian manifold with $-1 \leq \sec_g \leq \kappa$, $n > \kappa + 1$, and*

$$-Ric \geq \delta g, \quad (n-1) \geq \delta > 0.$$

For any $p \in M$ and geodesic sphere $S_r(p)$ with $r \leq \gamma_g(M)$, we have

$$H(r, \sigma) \geq \sqrt{\delta} \coth(\sqrt{\delta} r), \quad \forall \sigma \in S_r(p).$$

Proof. On any common interval of definition for H and u , equations (29) and (30) imply the following first inequality:

$$((u - H)e^{\int (u+H)})' \leq 0.$$

Thus, if we can find r_0 such that $H(r_0, \sigma) \geq u(r_0)$, we then have that $H(r, \sigma) \geq u(r)$ for any $r \geq r_0$. Now, recall from equation (25) that

$$(n-1) \cot(\kappa r) \leq H(r, \sigma) \leq (n-1) \coth(r), \quad (32)$$

for $r \leq \max(\gamma_g(M), \frac{\pi}{2\kappa})$. Thus, for r' sufficiently close to zero, we have $H(r', \sigma) \geq \sqrt{\delta} \coth(\sqrt{\delta} r')$, $\forall \sigma \in S_r$, and the result follows. \square

4 An Integral Inequality for Harmonic Forms

The main result of this section is the inequality given in Theorem 38. This inequality controls the L^2 -norm of a harmonic k -form on a ball of fixed radius, in terms of the L^2 -norm on the complement of the given ball. We begin with a lemma giving pointwise and integral bounds on the geometric quantity $q(r)$ appearing in the Price equality given in Proposition 16. Key to this lemma are the Riccati-Rauch arguments of Section 3.

Lemma 33. *Let (M^n, g) be a Riemannian manifold with $-1 \leq \sec_g \leq \kappa$ and $0 \leq \kappa \leq 1$. Let α be a harmonic k -form on M . Assume further that*

$$-Ric \geq \delta g, \quad (n-1) \geq \delta > 4k^2.$$

Then for $0 \leq \sigma < r < \gamma_g(M)$,

$$\int_{\sigma}^r q(s) ds \geq (r - \sigma) \left(\frac{\sqrt{\delta}}{2} - k \right) + k \ln(1 - e^{-2\sigma}). \quad (34)$$

Define $r_0 := \frac{1}{\kappa} \operatorname{arccot}(\frac{\sqrt{\delta}}{n-1}) \in (0, \pi/2\kappa)$. If

$$\frac{\sqrt{\delta}}{2k} \geq \coth(r_0) + \frac{\epsilon}{k}$$

then

$$\epsilon \leq q(r) \leq (n-1) \coth(r), \quad (35)$$

for any $p \in M$ and $r \leq \gamma_g(M)$.

Proof. Recall equation (24) gives the upper bound on the second fundamental form h of the geodesic sphere S_r :

$$h \leq \coth(r) g_r,$$

for all $r \leq \gamma_g(M)$. Then Proposition 21 yields

$$q(r) \geq \frac{H}{2} - k \coth(r) + \mu(r) \coth(r) \geq \frac{H}{2} - k \coth(r).$$

The lower bound on h in (24) and the monotonicity of $\frac{\cot(\kappa r)}{\coth(r)}$ on $(0, \pi/2\kappa)$, implies that given $r_0 < \frac{\pi}{2\kappa}$ for any $r \in (0, r_0]$ we have

$$\begin{aligned} \frac{H(r, \sigma)}{2} - k \coth(r) &\geq \frac{(n-1) \cot(\kappa r)}{2} - k \coth(r) \\ &\geq k \coth(r_0) \left(\frac{(n-1) \cot(\kappa r_0)}{2k \coth(r_0)} - 1 \right) \geq \epsilon. \end{aligned} \quad (36)$$

On the other hand by Lemma 31 and the monotonicity of $\coth(r)$ on $(0, \infty)$, we have for all $r > 0$,

$$\begin{aligned} \frac{H(r, \sigma)}{2} - k \coth(r) &\geq \frac{\sqrt{\delta}}{2} \coth(\sqrt{\delta}r) - k \coth(r) \\ &\geq k \coth(r_0) \left(\frac{\sqrt{\delta}}{2k \coth(r_0)} - 1 \right) \geq \epsilon. \end{aligned} \quad (37)$$

Expanding $\coth(t) = 1 + O(e^{-2t})$ as a power series in e^{-2t} and estimating error terms, the first line (37) implies (34). The last line combined with (36) implies the lower bound in (35). The upper bound is a simple consequence of (32). \square

We are now ready to prove the main technical result of this section.

Theorem 38. *Let (M^n, g) be a Riemannian manifold with $-1 \leq \sec_g \leq \kappa$, $0 \leq \kappa \leq 1$ and dimension $n \geq 6$. Let α be a harmonic k -form. Assume*

$$-Ric \geq \delta g, \quad \delta > 4k^2.$$

Let again $r_0 = \frac{1}{\kappa} \operatorname{arccot}(\frac{\sqrt{\delta}}{n-1})$. Let $C_{\sigma,k} = \coth(\sigma)^{2k}$. If for some $\epsilon > 0$,

$$\frac{\sqrt{\delta}}{2k} \geq \coth(r_0) + \frac{\epsilon}{k}, \quad (39)$$

then for any $0 < \sigma < \tau \leq \gamma_g(M)$ and $p \in M$,

$$\int_{B_\sigma(p)} |\alpha|^2 dv \leq \frac{(n-1)C_{\sigma,k}e^{-2(\tau-\sigma)(\frac{\sqrt{\delta}}{2}-k)}}{\epsilon(1-C_{\sigma,k}e^{-2(\tau-\sigma)(\frac{\sqrt{\delta}}{2}-k)})} \int_{B_\tau(p) \setminus B_\sigma(p)} |\alpha|^2 dv. \quad (40)$$

Proof. By Lemma 33, $q(r) \geq \epsilon > 0$. Equation (13) then implies $1 - 2\mu(s) > 0$ for $s \in [0, \gamma_g(M)]$. Then the weight function ϕ defined in (15) satisfies

$$\phi(r) \leq e^{-2 \int_\sigma^r q(s) ds} \leq C_{\sigma,k} e^{-2(r-\sigma)(\frac{\sqrt{\delta}}{2}-k)}, \quad (41)$$

where we have used (34) for the last inequality, and we set $C_{\sigma,k} = \coth(\sigma)^{2k}$ (not sharp). Inserting this inequality into (17) yields

$$\begin{aligned} &(1 - C_{\sigma,k}e^{-2(\tau-\sigma)(\frac{\sqrt{\delta}}{2}-k)}) \int_{B_\sigma(p)} q(r) |\alpha|^2 dv \\ &\leq C_{\sigma,k}e^{-2(\tau-\sigma)(\frac{\sqrt{\delta}}{2}-k)} \int_{B_\tau(p) \setminus B_\sigma(p)} q(r) |\alpha|^2 dv. \end{aligned} \quad (42)$$

The pointwise bounds on $q(r)$ given in Lemma 33 now yield

$$\begin{aligned} &(1 - C_{\sigma,k}e^{-2(\tau-\sigma)(\frac{\sqrt{\delta}}{2}-k)}) \epsilon \int_{B_\sigma(p)} |\alpha|^2 dv \\ &\leq C_{\sigma,k}e^{-2(\tau-\sigma)(\frac{\sqrt{\delta}}{2}-k)} (n-1) \coth(\sigma) \int_{B_\tau(p) \setminus B_\sigma(p)} |\alpha|^2 dv, \end{aligned} \quad (43)$$

which then concludes the proof. \square

5 From Integral Inequalities to Dimension Estimates

In this section, we use the integral inequalities of the previous section to extract dimension estimates. In the following section, we will sharpen these estimates using cohomological techniques. We include the slightly weaker results here for the reader who may be interested in applying them to related problems - such as bounding the dimensions of eigenspaces with very small eigenvalue - to which these techniques, but not their cohomological improvement, readily apply.

In order to extract dimension estimates from (40), we first need the following standard lemma.

Lemma 44. *Let (M^n, g) be a closed Riemannian manifold. There exists $\alpha \in \mathcal{H}_g^k(M)$, with $\|\alpha\|_{L^2} = 1$, such that*

$$\max_{p \in M} |\alpha|^2 \geq \frac{k!(n-k)!b_k(M)}{n!Vol(M)},$$

where $b_k(M) = \dim_{\mathbb{R}} \mathcal{H}_g^k(M)$ is the k -th Betti number of M .

Proof. Let $K(x, y)$ denote the Schwartz kernel of the L^2 -orthogonal projection onto $\mathcal{H}_g^k(M)$. Then

$$\int_M \text{tr} K(x, x) dv = \dim \mathcal{H}_g^k(M). \quad (45)$$

Hence, there exists $p \in M$ such that

$$\text{tr} K(p, p) \geq \frac{b_k(M)}{Vol(M)}. \quad (46)$$

Then there exists a unit eigenvector z of $K(p, p)$ with eigenvalue $\lambda \geq \frac{k!(n-k)!b_k(M)}{n!vol(M)}$. Since K is the Schwartz kernel of an L^2 -orthogonal projection,

$$\|K(x, p)z\|_{L^2}^2 = \langle K(p, p)z, z \rangle = \lambda. \quad (47)$$

Set

$$\alpha(x) := \frac{K(x, p)z}{\sqrt{\lambda}}. \quad (48)$$

Then $\|\alpha\|_{L^2} = 1$, and

$$|\alpha(p)|^2 = \frac{|K(p, p)z|^2}{\lambda} = \lambda \geq \frac{k!(n-k)!b_k(M)}{n!Vol(M)}. \quad (49)$$

□

The following lemma allows us to pass from integral inequalities to pointwise estimates.

Lemma 50. *Let (M^n, g) be a closed Riemannian manifold with*

$$-1 \leq \sec_g \leq 1.$$

Given a harmonic k -form $\alpha \in \mathcal{H}_g^k(M)$, for any $p \in M$ and $R < \min(\gamma_g(M), 1)$ there exists a strictly positive constant $d(n, R) := d(n)(1 + \frac{1}{R})^n$ such that

$$\|\alpha\|_{L^\infty(B_{\frac{R}{2}}(p))}^2 \leq d(n, R) \|\alpha\|_{L^2(B_R(p))}^2.$$

Proof. This is a standard application of Moser iteration. \square

Theorem 51. *Let (M^n, g) be a Riemannian manifold with $-1 \leq \sec_g \leq \kappa$. Assume*

$$-\text{Ric} \geq \delta g, \quad \delta \geq 4k^2.$$

Let $r_0 = \frac{1}{\kappa} \text{arccot}(\frac{\sqrt{\delta}}{n-1})$. If

$$\frac{\sqrt{\delta}}{2k} \geq \coth(r_0) + \frac{\epsilon}{k}, \quad (52)$$

then there exists a positive constant $c(n)$ depending on the dimension only such that for $\gamma_g(M)$ large

$$\frac{b_k(M)}{\text{Vol}_g(M)} \leq c(n) \epsilon^{-1} e^{-2\gamma_g(M)(\frac{\sqrt{\delta}}{2} - k)}.$$

Proof. By Lemma 44, there exists $\alpha \in \mathcal{H}_g^k(M)$ and $p \in M$, with $\|\alpha\|_{L^2(M)} = 1$, and

$$|\alpha(p)|^2 \geq \frac{k!(n-k)!b_k(M)}{n!\text{Vol}(M)}. \quad (53)$$

On the other hand, Lemma 50 and equation (40) yield

$$|\alpha|^2(p) \leq d(n, 1/2) \|\alpha\|_{L^2(B_p(1))}^2 \leq \frac{\tilde{c}(n)}{\epsilon} e^{-2\gamma_g(M)-1)(\frac{\sqrt{\delta}}{2}-k)} \|\alpha\|_{L^2(B_p(\tau))}^2, \quad (54)$$

where $d(n, 1/2) > 0$ is the constant of Lemma 50, $\tilde{c}(n) := 2d(n, 1/2)(n-1)C_{1,k+\frac{1}{2}}$, and $\gamma_g(M)$ is assumed so large that $1 - C_{1,k}e^{-2(\gamma_g(M)-1)(\frac{\sqrt{\delta}}{2}-k)} \geq \frac{1}{2}$. \square

6 Excising Geodesic Balls

In Section 5, we estimate $q(r)$ from below by combining Rauch's comparison with a Riccati type argument. Under the curvature constraint $\frac{\sqrt{\delta}}{2} > k \coth(r_0)$, with $r_0 := \frac{1}{\kappa} \text{arccot}(\frac{\sqrt{\delta}}{n-1})$, these arguments suffice to provide a positive uniform lower bound on $q(r)$ for any r . The deleterious effects of positive curvature

in the comparison arguments diminish at large radius. In fact, it only effects Rauch's comparison for small values of r . On the other hand, under our usual Ricci curvature assumption, the positive curvature does not affect the Riccati argument for large values of r . Thus it is natural to work in the complement of a large ball. In this section, we use cohomological arguments to remove the application of Rauch's comparison for small values of r . We begin with the following reformulation of Equation (13) with two boundary components.

Proposition 55. *Let (M^n, g) be complete and satisfy $-1 \leq \sec_g$. Given a point $p \in M$, $\alpha \in \mathcal{H}_g^k(M)$, and $\sigma \leq \tau \leq \gamma_g(M)$, we have*

$$\begin{aligned} \int_{S_\tau(p)} \left(\frac{1}{2} - \mu(\tau) \right) |\alpha|^2 d\sigma &= \int_{S_\sigma(p)} \left(\frac{1}{2} - \mu(\sigma) \right) |\alpha|^2 d\sigma \\ &+ \int_{B_\tau(p) \setminus B_\sigma(p)} q(r) |\alpha|^2 dv. \end{aligned} \quad (56)$$

Moreover, with ϕ as defined in (15),

$$\phi(\tau) \int_{B_\tau(p) \setminus B_\sigma(p)} q(r) |\alpha|^2 dv = (1 - \phi(\tau)) \int_{S_\sigma(p)} \left(\frac{1}{2} - \mu(\sigma) \right) |\alpha|^2 d\sigma. \quad (57)$$

Proof. This follows from the same arguments as (13) and (17). \square

In Section 5, we showed that if ϕ decays to zero, it is possible to give effective estimates on the size of Betti numbers normalized by the Riemannian volume. In order to control the size of ϕ , it is necessary to understand the sign not only of $q(r)$ but also of $\frac{1}{2} - \mu(r)$, see Equation (15). In particular, we require $\mu(r) < \frac{1}{2}$ for all r sufficiently large. This inequality follows from (56), if $\mu(\sigma) < \frac{1}{2}$. Thus, it is natural to work with harmonic forms with Neumann boundary condition.

Given a complete Riemannian manifold with boundary, Ω , let $\mathcal{H}_{2,N}^k(\Omega)$ denote the L^2 -harmonic k -forms on Ω satisfying Neumann boundary conditions on $\partial\Omega$. Let $H_{2,N}^k(\Omega)$ denote the (absolute) L^2 -cohomology of Ω . It is easy to see that, if 0 is not in the essential spectrum of Δ_k , then

$$\mathcal{H}_{2,N}^k(\Omega) \simeq H_{2,N}^k(\Omega). \quad (58)$$

For more details on L^2 -cohomology we refer to Section 8.

The next lemma is the natural analogue of Theorem 38 for Neumann harmonic forms.

Lemma 59. *Let (M^n, g) be a complete Riemannian manifold with $-1 \leq \sec_g \leq 1$. Let $p \in M$, $\rho < \gamma_g(M)$, and $\alpha \in \mathcal{H}_{2,N}^k(M \setminus B_\rho(p))$. Assume there exists $\delta > 4k^2$ and $\epsilon > 0$ such that*

$$-Ric \geq \delta g,$$

and

$$\frac{\sqrt{\delta}}{2} \coth(\sqrt{\delta}\rho) - k \coth(\rho) \geq \epsilon. \quad (60)$$

Then for any $\sigma \in (\rho, \gamma_g(M))$, $\tau \in (\sigma, \gamma_g(M))$ and $\sigma < R < \tau$,

$$\int_{B_R(p) \setminus B_\sigma(p)} |\alpha|^2 dv \leq \frac{n-1}{\epsilon} \coth(\sigma)^{2k+1} e^{-(\sqrt{\delta}-2k)(\tau-R)} \int_{B_\tau(p) \setminus B_\sigma(p)} |\alpha|^2 dv. \quad (61)$$

Proof. Recall that the Neumann boundary condition implies $i_{\partial_r} \alpha = 0$ on $S_\rho(p)$. Equivalently, $\mu(\rho) = 0$. Hence (56) with $\sigma = \rho$ implies $\mu(r) \leq \frac{1}{2}$, $\forall r \in [\rho, \gamma_g(M)]$, as long as $q(r) \geq 0$, for $\forall r \in [\rho, \gamma_g(M)]$. Taking (57) for two different values of τ and then taking the difference of the two equations gives

$$\begin{aligned} & \phi(\tau) \int_{B_\tau(p) \setminus B_\sigma(p)} q(r) |\alpha|^2 dv \\ &= \phi(R) \int_{B_R(p) \setminus B_\sigma(p)} q(r) |\alpha|^2 dv + (\phi(R) - \phi(\tau)) \int_{S_\sigma(p)} \left(\frac{1}{2} - \mu(\sigma)\right) |\alpha|^2 d\sigma. \end{aligned} \quad (62)$$

The hypotheses imply ϕ is monotonically decreasing. Our curvature estimates imply $q(s) > \frac{\sqrt{\delta}}{2} - k \coth(s)$, and the desired estimate follows from (15). \square

The Price inequality of Lemma 59 can now be applied to derive a vanishing for certain spaces of Neumann harmonic forms.

Corollary 63. *Let (M^n, g) be a simply connected non-compact complete Riemannian manifold without conjugate points and $-1 \leq \sec_g \leq 1$. Assume there exists $\delta > 4k^2 > 0$ such that*

$$-Ric \geq \delta g.$$

Let $p \in M$. Then for ρ sufficiently large, $\mathcal{H}_{2,N}^k(M \setminus B_\rho(p)) = 0$.

Proof. Taking $\tau \rightarrow \infty$ in the preceding lemma, we see that α vanishes identically. \square

Next, we consider closed manifolds without conjugate points.

Proposition 64. *Let (M^n, g) be a closed manifold without conjugate points with $-1 \leq \sec_g \leq 1$. Assume there exists $\delta > 4k^2$ such that*

$$-Ric \geq \delta g.$$

Let $\pi_i : M_i \rightarrow M$ be a sequence of Riemannian covers of M with $\gamma_g(M_i) \rightarrow \infty$. Let $h_i \in \mathcal{H}^k(M_i)$, and $p_i \in M_i$. Then there exists $c(n, k, \delta) > 0$ so that for $\gamma_{g_i}(M_i)$ sufficiently large,

$$\int_{B_\rho(p_i)} |h_i|^2 dv \leq c(n, k, \delta) e^{-(\sqrt{\delta}-2k)\gamma_g(M_i)} \|h_i\|^2. \quad (65)$$

Proof. Choose ρ sufficiently large so that $\frac{\sqrt{\delta}}{2} \coth(\sqrt{\delta}\rho) - k \coth(\rho) := \epsilon > 0$. Consider i sufficiently large so that $\rho < \gamma_g(M_i)$. Then for $k < n-1$, we have

$$\mathcal{H}^k(M_i) \simeq H^k(M_i) \simeq H_N^k(M_i \setminus B_\rho(p_i)) \simeq \mathcal{H}_N^k(M_i \setminus B_\rho(p_i)).$$

Consider the map from $\mathcal{H}^k(M_i)$ obtained as follows. Given $h \in \mathcal{H}^k(M_i)$, let $J_1(h)$ denote the $H_N^k(M_i \setminus B_\rho(p_i))$ harmonic representative of the restriction of h to $M_i \setminus B_\rho(p_i)$. Let b denote a coexact primitive for the restriction of $J_1(h)$ to $B_{\rho+2}(p_i) \setminus B_\rho(p_i)$. Let η be a smooth cutoff function with $|d\eta| < 2$, supported $B_{\rho+2}(p_i) \setminus B_\rho(p_i)$, identically zero near $B_{\rho+2}(p_i)$ and identically one in $B_{\rho+1}(p_i)$. Then $J_2(h) := J_1(h) - d(\eta b)$ defines an element in $H^k(M_i)$. Because every cycle in $H_k(M_i)$ has a representative disjoint from $B_\rho(p_i)$ and the integral of $J_2(h)$ over every such cycle is equal to the integral of h over the cycle, $J_2(h)$ is cohomologous to h . Let now $J_3(h)$ denote the M_i harmonic projection of $J_2(h)$. Then we have shown $J_3(h) = h$. Restriction and harmonic projection are norm nonincreasing. In particular, we have

$$\|J_1(h)\|_{L^2}^2 \leq \|h\|^2 - \int_{B_\rho} |h|^2 dv. \quad (66)$$

On the other hand, we have

$$\|h\|_{L^2}^2 = \|J_3(h)\|_{L^2}^2 \leq \|J_2(h)\|_{L^2}^2 \leq \|h\|^2 - \int_{B_\rho} |h|^2 dv + \|d(\eta b)\|_{L^2}^2. \quad (67)$$

Hence

$$\begin{aligned} \int_{B_\rho} |h|^2 dv &\leq \|d(\eta b)\|_{L^2}^2 \leq 2\|\eta J_1(h)\|_{L^2}^2 + 2\|d\eta \wedge b\|_{L^2}^2 \\ &\leq 2\|J_1(h)\|_{L^2(B_{\rho+2} \setminus B_\rho, g)}^2 + 8\|b\|_{L^2(B_{\rho+2} \setminus B_\rho, g)}^2. \end{aligned} \quad (68)$$

Use normal coordinates to fix a diffeomorphism $\zeta : B_{\rho+2}(p_i) \setminus B_\rho(p_i) \rightarrow S^{n-1} \times [0, 2]$. Write $H := \zeta^* J_1(h) = H_0 + dt \wedge H_1$, with $i_{\frac{\partial}{\partial t}} H_j = 0$, t the coordinate on $[0, 2]$. Fix $L \in [0, 2]$ minimizing $\int_{S_L} |H|^2 d\sigma$. Set

$$\beta_L(r) := \int_L^r H_1(s) ds. \quad (69)$$

Then since $d(\zeta^* J_1(h)) = 0$

$$d\beta_L = H - H_0(L).$$

In the product metric δ on $S^{n-1} \times [0, 2]$, we have

$$\begin{aligned} \|\beta_L\|_{L^2(S^{n-1} \times [0, 2], \delta)}^2 &= \int_0^2 \int_{S^{n-1}} \left(\int_L^r H_1(s) ds \right)^2 d\sigma dr \leq 2 \int_{S^{n-1} \times [0, 2]} |H_1|^2 dv \\ &\leq 2\|H\|_{L^2(S^{n-1} \times [0, 2], \delta)}^2. \end{aligned} \quad (70)$$

Let ξ denote a coexact primitive for $H_0(L)$ viewed as an exact form on S^{n-1} . Then in the product metric,

$$\|\xi\|_{L^2(S^{n-1} \times [0, 2], \delta)}^2 \leq W_n \|H\|_{L^2(S^{n-1} \times [0, 2], \delta)}^2,$$

with $W_n = O(\lambda_{1,k}^{-1})$, where $\lambda_{1,k}$ denotes the first eigenvalue of the Laplace Beltrami operator for k -forms on the sphere. (Our hypotheses imply $k \neq 0, n-1$.) By construction,

$$d(\beta_L + \xi) = H.$$

There exists $C_1(\rho) > 0$ so that

$$C_1^{-1} \|\cdot\|_{L^2(B_{\rho+2}(p) \setminus B_\rho(p), g)}^2 \leq \|\cdot\|_{L^2(S^{n-1} \times [0, 2], \delta)}^2 \leq C_1 \|\cdot\|_{L^2(B_{\rho+2}(p) \setminus B_\rho(p), g)}^2. \quad (71)$$

Then we have

$$\|\beta_L + \xi\|_{L^2(S^{n-1} \times [0, 2], \delta)}^2 \leq (W_n + 2\sqrt{2}\sqrt{W_n} + 2) \|H\|_{L^2(S^{n-1} \times [0, 2], \delta)}^2. \quad (72)$$

Equation (72) combined with (68) and (71) gives

$$\int_{B_\rho} |h|^2 dv \leq d(W_n, C_1) \|J_1(h)\|_{L^2(B_{\rho+2}(p) \setminus B_\rho(p), g)}^2, \quad (73)$$

where $d(W_n, C_1)$ is a positive constant depending on W_n and C_1 . Using (61), we have

$$\int_{B_\rho} |h|^2 dv \leq d(W_n, C_1) \frac{n-1}{\epsilon} \coth(\rho)^{2k+1} e^{-(\sqrt{\delta}-2k)(\tau-\rho-2)} \int_{B_\tau \setminus B_\rho(p)} |J_1(h)|^2 dv, \quad (74)$$

and the result follows. \square

We can now prove the main result of this section.

Corollary 75. *Let (M^n, g) be a closed Riemannian manifold without conjugate points and with $-1 \leq \sec_g \leq 1$. Assume there exists $\delta > 4k^2 > 0$ such that*

$$-\text{Ric} \geq \delta g.$$

Let $\pi_i : M_i \rightarrow M$ be a sequence of Riemannian covers of M with $\gamma_g(M_i) \rightarrow \infty$. Then there exists $b(n, k) > 0$ so that for $\gamma_g(M_i)$ sufficiently large,

$$\frac{b_k(M_i)}{\text{Vol}(M_i)} \leq b(n, k, \delta) e^{-(\sqrt{\delta}-2k)\gamma_{g_i}(M_i)}. \quad (76)$$

In particular,

$$\lim_{i \rightarrow \infty} \frac{b_k(M_i)}{\text{Vol}(M_i)} = 0. \quad (77)$$

Proof. Apply Lemmas 44 and 50 to Proposition 64. \square

Congruence subgroups of arithmetic groups of \mathbb{Q} rank 0 algebraic groups provide an important and widely studied class of examples of towers of covers. These lattices have large injectivity radii relative to their covolumes with respect to the natural locally symmetric metric. For more details see [SX91], [Yeu94], and [Mar14]. The following definition abstracts the injectivity radius properties of such lattices.

Definition 78. Let (M^n, g) be a closed Riemannian manifold with infinite residually finite fundamental group $\Gamma := \pi_1(M^n)$. Given a cofinal filtration $\{\Gamma_i\}$ of Γ , for any index i let us set

$$r_i := \inf\{d_{\tilde{g}}(z, \gamma_i z)/2 \mid z \in \tilde{X}, \gamma_i \in \Gamma_i, \gamma_i \neq 1\}$$

where (\tilde{X}, \tilde{g}) is the Riemannian universal cover. We say that the cofinal filtration is of **congruence type** if there exist constants, $0 < \alpha(n, \{\Gamma_i\}) < 1$, $g(n, \{\Gamma_i\}) > 0$ such that

$$e^{r_i} \geq g(n, \{\Gamma_i\})[\Gamma : \Gamma_i]^{\alpha(n, \{\Gamma_i\})}$$

for any $i \geq 1$, where $[\Gamma : \Gamma_i]$ is the index of Γ_i in Γ . We call the constant α the **exponent** of the cofinal filtration.

We restate our Betti number asymptotics under the additional hypothesis of a congruence type cofinal filtration with exponent α .

Corollary 79. *Let (M^n, g) be a Riemannian manifold without conjugate points with $-1 \leq \sec_g \leq \kappa$ and residually finite fundamental group Γ . Assume*

$$-Ric \geq \delta g, \quad \delta > 4k^2.$$

Let $\{\Gamma_i\}_i$ be a congruence type cofinal filtration of Γ of exponent α . Denote by $\pi_i : M_i \rightarrow M$ the regular Riemannian cover of M associated to Γ_i . Then

$$b_k(M_i) \leq d(n, k, \delta, \Gamma) Vol(M_i)^{1-2\alpha(\frac{\sqrt{\delta}}{2}-k)}.$$

where $d(n, k, \Gamma)$ is a positive constant.

Proof. Since (M^n, g) has no conjugate points, for any $i \geq 1$ the numerical invariant $r_i > 0$ given in Definition 78 is simply the injectivity radius $\gamma_{g_i}(M_i)$. Since

$$Vol(M_i) = Vol(M)[\Gamma : \Gamma_i],$$

by Corollary 75 we have

$$b_k(M_i) \leq \frac{b(n, k)}{e^{2r_i(\frac{\sqrt{\delta}}{2}-k)}} Vol(M_i) \leq \frac{b(n, k) Vol(M)}{g(n, \Gamma)^{2(\frac{\sqrt{\delta}}{2}-k)}} [\Gamma : \Gamma_i]^{1-2(\frac{\sqrt{\delta}}{2}-k)\alpha},$$

and the proof is complete. \square

7 Inequalities for Negatively Pinched Manifolds in Dimensions $n \geq 4$

Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 4$ such that

$$-b^2 \leq \sec_g \leq -a^2, \tag{80}$$

for some $a, b \in (0, \infty)$. Let $p \in M$, and denote by B_R the ball of radius $0 < R < \gamma_g(M)$ centered at p . In B_R we write the metric in geodesic spherical coordinates as $g = dr^2 + g_r$. With the curvature assumptions (80), Rauch's comparison (see Theorem 22) gives the following two sided bound on the second fundamental form of any geodesic sphere:

$$a \coth(ar) \leq \text{Hess}(r, u) \leq b \coth(br), \quad (81)$$

for any $r \leq \gamma_g(M)$ and for any point u on the geodesic sphere S_r . Taking the trace of (81) gives the corresponding mean curvature bound:

$$(n-1)a \coth(ar) \leq H(r, u) \leq (n-1)b \coth(br), \quad (82)$$

for any $u \in S_r(p)$, given any $p \in M$. This implies that for any $r \leq \gamma_g(M)$, we have

$$\frac{(n-1)}{2}a \coth(ar) - kb \coth(br) \leq q(r) \leq \frac{(n-1)}{2}b \coth(br) - ka \coth(ar). \quad (83)$$

We now specialize our Price inequalities to harmonic forms on closed pinched negatively curved manifolds of dimension $n \geq 4$. This is the main result of this section.

Theorem 84. *Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 4$. Assume the sectional curvature is ϵ -pinched :*

$$-(1+\epsilon)^2 \leq \sec_g \leq -1,$$

with $\epsilon \geq 0$. Let k be a non-negative integer such that

$$\epsilon_{n,k} := (n-1) - 2k(1+\epsilon) > 0.$$

For $\alpha \in \mathcal{H}_g^k(M)$, and $0 < \sigma < \tau \leq \gamma_g(M)$, we have

$$\int_{B_\sigma(p)} |\alpha|^2 dv \leq \frac{e^{-(r-\sigma)\epsilon_{n,k}}}{1 - e^{-(r-\sigma)\epsilon_{n,k}}} \frac{(n-1)}{\epsilon_{n,k}} (1+\epsilon) \coth(\sigma) \int_{B_\tau(p) \setminus B_\sigma(p)} |\alpha|^2 dv. \quad (85)$$

Proof. Recall the Price equality given in (17)

$$\int_{B_\sigma(p)} q(r) |\alpha|^2 dv = \phi(\tau) \int_{B_\tau(p)} q(r) |\alpha|^2 dv. \quad (86)$$

By (83) we have

$$q(r) \geq \frac{(n-1)}{2} \coth(r) - k(1+\epsilon) \coth((1+\epsilon)r) > \frac{(n-1)}{2} - k(1+\epsilon). \quad (87)$$

Hence by (15), we have

$$\phi(r) \leq e^{-(r-\sigma)(n-1-2k(1+\epsilon))}, \quad (88)$$

which combined with (17) gives

$$\int_{B_\sigma(p)} q(r) |\alpha|^2 dv \leq \frac{e^{-(r-\sigma)(n-1-2k(1+\epsilon))}}{1 - e^{-(r-\sigma)(n-1-2k(1+\epsilon))}} \int_{B_\tau(p) \setminus B_\sigma(p)} q(r) |\alpha|^2 dv. \quad (89)$$

The upper bound of Equation (83) implies

$$q(r) \leq \frac{(n-1)}{2} (1+\epsilon) \coth((1+\epsilon)r) - k \coth(r),$$

so that for any $r \geq \sigma$, we have

$$q(r) \leq \frac{n-1}{2} (1+\epsilon) \coth(r). \quad (90)$$

The lower bound on q given in (83) combined with (90) then yields

$$\int_{B_\sigma(p)} |\alpha|^2 dv \leq \frac{e^{-(r-\sigma)\epsilon_{n,k}}}{1 - e^{-(r-\sigma)\epsilon_{n,k}}} \frac{(n-1)}{\epsilon_{n,k}} (1+\epsilon) \coth(\sigma) \int_{B_\tau(p) \setminus B_\sigma(p)} |\alpha|^2 dv, \quad (91)$$

as claimed. \square

Corollary 92. *Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 4$. Assume*

$$-(1+\epsilon)^2 \leq \sec_g \leq -1$$

with $\epsilon \geq 0$. There exists a constant $c_{n,k} > 0$ so that for each non-negative integer k such that

$$\epsilon_{n,k} := (n-1) - 2k(1+\epsilon) > 0,$$

$$\frac{b_k(M)}{\text{Vol}_g(M)} \leq c(n,k) e^{-\epsilon_{n,k} \gamma_g(M)},$$

for $\gamma_g(M) > 1 + \frac{\ln(2)}{\epsilon_{n,k}}$.

Proof. Apply Lemmas 44 and 50 to Proposition 84. The (nonsharp) constraint on $\gamma_g(M)$ serves to control denominators. \square

Finally, we study the borderline case where $\epsilon_{n,k} = 0$ and $\epsilon > 0$.

Theorem 93. *Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 4$. Assume the sectional curvature is ϵ -pinched :*

$$-(1+\epsilon)^2 \leq \sec_g \leq -1,$$

with $\epsilon > 0$. Let k be a positive integer such that

$$\epsilon_{n,k} := (n-1) - 2k(1+\epsilon) = 0.$$

For $\alpha \in \mathcal{H}_g^k(M)$ and $0 < 1 \leq \tau \leq \gamma_g(M)$, we have

$$\int_{B_1(p)} |\alpha|^2 dv \leq \frac{\sinh^2(1+\epsilon)}{2k(1+\epsilon)\epsilon(\tau-1)} \int_{B_\tau(p) \setminus B_1(p)} |\alpha|^2 dv. \quad (94)$$

Proof. Given the assumption that $\epsilon_{n,k} = 0$, we write the first inequality of (87) as

$$q(r) \geq k(1+\epsilon)(\coth(r) - \coth((1+\epsilon)r)).$$

Using the mean value theorem, we may estimate the lower bound by

$$q(r) \geq \frac{k(1+\epsilon)r\epsilon}{\sinh^2(r+\epsilon r)}. \quad (95)$$

The righthand side of this inequality is monotonically decreasing in r . Hence for $r \in [0, 1]$ we have

$$q(r) \geq \frac{k(1+\epsilon)\epsilon}{\sinh^2(1+\epsilon)}. \quad (96)$$

Recall (17) and (13): for $1 < \tau \leq \gamma_g(M)$,

$$\int_{B_\sigma(p)} q(r) |\alpha|^2 dv = \phi(\tau) \int_{B_\tau(p)} q(r) |\alpha|^2 dv = \phi(\tau) \int_{S_\tau(p)} \left(\frac{1}{2} - \mu(\tau)\right) |\alpha|^2 d\sigma. \quad (97)$$

When $\epsilon_{n,k} = 0$, ϕ need not decay exponentially, but by definition (15),

$$0 < \phi(r) \leq 1. \quad (98)$$

Thus, for any $s \geq 1$

$$\begin{aligned} \frac{k(1+\epsilon)\epsilon}{\sinh^2(1+\epsilon)} \int_{B_1(p)} |\alpha|^2 dv &\leq \int_{B_1(p)} q(r) |\alpha|^2 dv \\ &\leq 1 \cdot \int_{B_s(p)} q(r) |\alpha|^2 dv \\ &= \int_{S_s(p)} \left(\frac{1}{2} - \mu(s)\right) |\alpha|^2 d\sigma. \end{aligned} \quad (99)$$

Integrating Equation (99) from 1 to $1 < \tau \leq \gamma_g(M)$, yields

$$\frac{k(1+\epsilon)\epsilon}{\sinh^2(1+\epsilon)} (\tau-1) \int_{B_1(p)} |\alpha|^2 dv \leq \frac{1}{2} \int_{B_\tau(p) \setminus B_1(p)} |\alpha|^2 dv,$$

which concludes the proof. \square

Corollary 100. *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 4$. Assume*

$$-(1 + \epsilon)^2 \leq \sec_g \leq -1,$$

with $\epsilon > 0$ satisfying $k := \frac{n-1}{2(1+\epsilon)}$ is an integer. If $\gamma_g(M) > 1$, there exists a positive constant $c(n, k, \epsilon)$ such that

$$\frac{b_k(M)}{\text{Vol}_g(M)} \leq \frac{c(n, k, \epsilon)}{\gamma_g(M) - 1}.$$

Proof. Choose a normalized $\alpha \in \mathcal{H}_g^k(M)$ as in Lemma 44, and let $p \in M$ be a point where the norm of α achieves its maximum. Since $\|\alpha\|_{L^2} = 1$, for $1 < \tau \leq \gamma_g(M)$, Equation (94) combined with Lemma 44 and with Lemma 50 with $R = 1$ then gives

$$\begin{aligned} \frac{b_k(M)}{\text{Vol}(M)} &\leq \binom{n}{k} |\alpha|_p^2 \leq \binom{n}{k} \frac{d(n, 1) \sinh^2(1 + \epsilon)}{2k(1 + \epsilon)\epsilon(\tau - 1)} \|\alpha\|_{L^2(B_1)}^2 \\ &\leq \binom{n}{k} \frac{d(n, 1) \sinh^2(1 + \epsilon)}{2k(1 + \epsilon)\epsilon(\tau - 1)}. \end{aligned}$$

Take $\tau = \gamma_g(M)$ to complete the proof. \square

We can now study the asymptotic behavior of Betti numbers with respect to coverings of negatively pinched Riemannian manifolds.

Corollary 101. *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 4$ with residually finite fundamental group $\Gamma := \pi_1(M)$. Assume*

$$-(1 + \epsilon)^2 \leq \sec_g \leq -1,$$

with $\epsilon \geq 0$. Given a cofinal filtration $\{\Gamma_i\}$ of Γ , denote by $\pi_i : M_i \rightarrow M$ the regular Riemannian cover of M associated to Γ_i . For each non-negative integer such that $\epsilon_{n,k} = (n - 1) - 2k(1 + \epsilon) > 0$,

$$\lim_{i \rightarrow \infty} \frac{b_k(M_i)}{\text{Vol}(M_i)} = 0. \quad (102)$$

Moreover, if $\epsilon > 0$, the same holds true for k -forms of critical degree such that $\epsilon_{n,k} = 0$.

Proof. Given a cofinal filtration $\{\Gamma_i\}$ of Γ ,

$$\lim_{i \rightarrow \infty} \gamma_{g_i}(M_i) = \infty$$

where M_i is equipped with the pull back metric $g_i := \pi_i^*(g)$. For a proof, see Theorem 2.1. in [DW78]. The corollary is now a consequence of Corollary 92 if $\epsilon_{n,k} > 0$. If $\epsilon_{n,k} = 0$, we instead appeal to Corollary 100. \square

We now consider cofinal filtrations of congruence type (see Definition 78).

Corollary 103. *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 4$ with residually finite fundamental group $\Gamma := \pi_1(M)$. Assume*

$$-(1 + \epsilon)^2 \leq \sec_g \leq -1,$$

with $\epsilon \geq 0$. Given a cofinal filtration $\{\Gamma_i\}$ of Γ of congruence type with exponent $\alpha \in (0, 1)$, denote by $\pi_i : M_i \rightarrow M$ the regular Riemannian cover of M associated to Γ_i . For any integer k such that

$$\epsilon_{n,k} := (n - 1) - 2k(1 + \epsilon) > 0,$$

and for all i such that

$$\gamma_{g_i}(M_i) > 1 + \frac{\ln(2)}{\epsilon_{n,k}},$$

we have

$$b_k(M_i) \leq k(n, k, \epsilon, \Gamma) \text{Vol}(M_i)^{1 - \epsilon_{n,k}\alpha}.$$

where $k(n, k, \epsilon, \Gamma)$ is a positive constant. Moreover, if $\epsilon > 0$, then for k -forms of the critical degree such that $\epsilon_{n,k} = 0$ we have

$$b_k(M_i) \leq l(n, k, \epsilon, \Gamma) \frac{\text{Vol}_{g_i}(M_i)}{\alpha \ln(\text{Vol}_{g_i}(M_i))},$$

for some positive constant $l(n, k, \epsilon, \Gamma)$.

Proof. Re-express the functions of injectivity radius in Theorems 92 and 93 as functions of $\text{Vol}(M)$. \square

When $M = \mathbb{H}_{\mathbb{R}}^n / \Gamma$ is a compact hyperbolic manifold of dimension $n = 2k + 1$, we can also extend the results of Corollary 100 to estimate $b_k(M)$. In this case, in order to apply the usual Price inequality approach we need to understand the magnitude of μ .

Corollary 104. *Let M be a compact hyperbolic manifold of dimension $2k + 1$ with $\gamma_g(M) > 1$. There exists $C_k > 0$ such that*

$$b_k(M) \leq C_k \frac{\text{Vol}(M)}{\gamma_g(M) - 1}. \quad (105)$$

Proof. The proof is the same as Corollary 100, except that we now have

$$q(r) = \mu(r) \coth(r),$$

so that we need to estimate the magnitude of μ from below. Let h be a harmonic k form, $k \neq 0, n$, not vanishing at $p \in M$. Recall that we have $\mu(0) = \frac{k}{2k+1}$, see Lemma 18. In hyperbolic geometry, it is possible to show there is a k dependent positive lower bound $\tilde{\mu}$ for μ on $[0, 1]$. This lower bound can be explicitly estimated using separation of variables. Relying less on the special geometry, we may also estimate μ as follows. Let $L \in (0, 1]$. Identify $B_R \setminus \{0\}$

with $(0, R) \times S^{n-1}$ in the usual manner, and write $h = h_0 + dr \wedge h_1$, with $i_{\partial_r} h_j = 0$. Use the product structure to identify the h_j with a one parameter family of forms on S^{n-1} . Set

$$b_L(r) = \int_L^r h_1(s) ds. \quad (106)$$

Then since $dh = 0$ we have

$$db_L(r) = h(r) - h_0(L).$$

The closed form $h_0(L)$ is exact on S^{n-1} . Hence there exists a $k-1$ form β_0 on S^{n-1} such that

$$d\beta_0 = h_0(L).$$

Moreover,

$$\int_{S_L} |\beta_0|^2 d\sigma \leq c_{n,k}^2 \sinh^2(L) \int_{S_L} |h_0|^2 d\sigma,$$

for some $c_{n,k} > 0$. On the other hand, we have

$$\begin{aligned} \int_{B_L(p)} |h|^2 dv &= \int_{B_L(p)} \langle d(b_L + \beta_0), h \rangle dv \\ &= \int_{S_L(p)} \langle \beta_0, e^*(dr)h \rangle d\sigma \leq \sqrt{\mu(L)} c_{n,k} \sinh(L) \|h\|_{S_L}^2 \\ &= \frac{2\sqrt{\mu(L)}}{1-2\mu(L)} c_{n,k} \sinh(L) \int_{B_L(p)} q(r) |h|^2 dv, \end{aligned} \quad (107)$$

where we have used (13) for the last equality. By Lemma 50

$$\begin{aligned} \sinh(L) \int_{B_{L/2}(p)} q(r) |h|^2 dv &\leq d(n) \left(1 + \frac{2}{L}\right)^n \sinh(L) \int_{B_L(p)} |h|^2 dv \int_{B_{L/2}(p)} q(r) dv \\ &\leq d(n, L) \int_{B_L(p)} |h|^2 dv. \end{aligned} \quad (108)$$

where

$$d(n, L) := d(n) \left(1 + \frac{2}{L}\right)^n \sinh(L) \text{Vol}(S^{n-1}) \sinh(L/2)^{n-1}.$$

On the other hand

$$\sinh(L) \int_{B_L \setminus B_{L/2}} q(r) |h|^2 dv \leq (n-1) \sinh(L) \coth(L/2) \int_{B_L} |h|^2 dv. \quad (109)$$

By combining Equations (108) and (109)

$$\sinh(L) \int_{B_L(p)} q(r) |h|^2 dv \leq C_1(n, L) \int_{B_L(p)} |h|^2 dv,$$

where $C_1(n, L)$ is a constant depending only on n and L . Inserting this estimate back into (107) gives

$$1 \leq \frac{2\sqrt{\mu(L)}}{1 - 2\mu(L)} c_{n,k} C_1(n, L). \quad (110)$$

Hence $\mu(L)$ is bounded below for L bounded. Given this lower bound, the proof proceeds exactly as in Corollary 100. \square

We summarize our Betti number estimates for real hyperbolic manifolds with the following corollary.

Corollary 111. *Let $X^n = \mathbb{H}_{\mathbb{R}}^n / \Gamma$ be a closed real hyperbolic manifold with $\sec_g = -1$ and injectivity radius $\gamma_g(X) > 1$. Given a cofinal filtration $\{\Gamma_i\}$ of Γ , let us denote by $\pi_i : X_i \rightarrow X$ the regular Riemannian cover of X associated to Γ_i . If $n = 2m$, for any integer $1 \leq k < m$, there exists a positive constant $c_1(n, k)$ such that*

$$\frac{b_k(X_i)}{\text{Vol}_{g_i}(X_i)} \leq c_1(n, k) e^{-(1+2(m-k))\gamma_{g_i}(X_i)}.$$

On the other hand, if $n = 2m + 1$, for any integer $k < m$ there exists a positive constant $c_2(n, k)$ such that

$$\frac{b_k(X_i)}{\text{Vol}_{g_i}(X_i)} \leq c_2(n, k) e^{-(2(m-k))\gamma_{g_i}(X_i)}.$$

In both cases, the sub volume growth of the Betti numbers along the tower of coverings is exponential in the injectivity radius. Finally, for $n = 2k + 1$ we have the existence of a positive constant $c_3(n, k)$ such that

$$\frac{b_k(M_i)}{\text{Vol}_{g_i}(M_i)} \leq \frac{c_3(n, k)}{\gamma_{g_i}(M_i) - 1}.$$

Proof. For $n = 2m$, notice that $\epsilon_{n,k} = \epsilon_{2m,m-1} = 1$. For $n = 2m + 1$, we have $\epsilon_{n,k} = \epsilon_{2m+1,m} = 0$. The statement in this case then follows from Corollary 92. For the critical case of k -forms in dimension $n = 2k + 1$, we use Corollary 104. \square

This result can also be obtained by trace formula techniques, but we consider our proof to be significantly simpler. For more details see again [Xue91], [SX91] and [Mar14].

8 L^2 -Cohomology and L^2 -Betti Numbers

Let $H_{2,N}^k(\Omega)$ denote the (absolute) L^2 -cohomology of Ω . Let Δ_k denote the Laplace Beltrami operator on k -forms. If 0 is not in the essential spectrum of Δ_k , then

$$\mathcal{H}_{2,N}^k(\Omega) \simeq H_2^k(\Omega). \quad (112)$$

Lemma 113. *Let (M^n, g) be a simply connected non-compact complete Riemannian manifold without conjugate points and $-1 \leq \sec_g \leq 1$. If there exists $\delta > 4k^2$ such that*

$$-Ric \geq \delta g,$$

then zero is not in the essential spectrum of Δ_k .

Proof. It is well-known that zero is not in the essential spectrum of Δ_k if and only if there is a compact set $K \subset M$ and a constant $\gamma > 0$ such that

$$\|d\alpha\|_{L^2}^2 + \|d^*\alpha\|_{L^2}^2 \geq \gamma \|\alpha\|_{L^2}^2,$$

for any smooth k -form α compactly supported in $M \setminus K$. (See for example [Ang93].) Fix $p \in M$. Choose ρ large enough so that $\left(\frac{\sqrt{\delta}}{2} - k \coth(\rho)\right) =: \epsilon$ is strictly positive. Given $\alpha \in C_c^\infty(\Lambda^k T^*(M \setminus \overline{B_\rho(p)}))$, choose R sufficiently large so that the support of α is contained in $B_R(p)$. Now, in the absence of the harmonicity assumption and with the addition of the support assumption, (13) becomes

$$\int_{B_R} (i_{-\partial_r} \alpha, d^* \alpha) dv + \int_{B_R} (i_{-\partial_r} d\alpha, \alpha) dv \geq \int_{B_R} q(r) |\alpha|^2 dv.$$

Since $|\partial_r| = 1$,

$$\int_{B_R} (i_{-\partial_r} \alpha, d^* \alpha) dv + \int_{B_R} (i_{-\partial_r} d\alpha, \alpha) dv \leq \left(\|d\alpha\|_{L^2}^2 + \|d^* \alpha\|_{L^2}^2\right)^{1/2} \|\alpha\|_{L^2}.$$

On the other hand, since the support of α does not intersect the closure of the ball $B_{r_\epsilon}(p)$ we have that

$$\int_{B_R(p)} q(r) |\alpha|^2 dv \geq \epsilon \int_{B_R(p)} |\alpha|^2 dv.$$

Setting $\gamma = \epsilon^2$ gives the desired lower bound on the spectrum. \square

Corollary 114. *Let (M^n, g) be a simply connected non-compact complete Riemannian manifold without conjugate points and $-1 \leq \sec_g \leq 1$. If there exists $\delta > 4k^2$ such that*

$$-Ric \geq \delta g,$$

then $\mathcal{H}_2^k(M) = H_2^k(M) = 0$.

Proof. By Lemma 113, zero is not in the essential spectrum of Δ_k . Thus, $\mathcal{H}_2^k(M) = H_2^k(M)$. The vanishing follows by applying Lemma 59 and standard long exact sequences. \square

We now collect some consequences of Corollary 114 regarding the vanishing of L^2 -Betti numbers of certain classes of manifolds without conjugate points and with negative Ricci curvature. The L^2 -Betti numbers are non-negative real valued numerical invariants associated to a closed Riemannian manifolds. They were originally introduced by Atiyah in [Ati76] in connection with L^2 -index theorems. Let us briefly recall their definition.

Definition 115. [Ati76, p. 44] Let (M^n, g) be a closed aspherical manifold. Let $\pi : (\tilde{M}, \pi^*(g)) \rightarrow (M, g)$ be the Riemannian universal cover. Thus, $M = \tilde{M}/\Gamma$ where Γ is a torsion free infinite group of isometries. The L^2 -Betti numbers of M are the von Neumann dimension of the Γ -module $\mathcal{H}_2^k(\tilde{M})$:

$$b_k^{(2)}(M) := \dim_{\Gamma}(\mathcal{H}_2^k(\tilde{M})).$$

We do not enter here into a detailed discussion of the theory of von Neumann dimension of Hilbert spaces with group actions. For our purposes, it suffices to recall (see Atiyah [Ati76], see also [Lüc02, Chapter I]) that

$$\dim_{\Gamma} \mathcal{H}_2^k(\tilde{M}) = 0 \quad \Leftrightarrow \quad \mathcal{H}_2^k(\tilde{M}) = 0. \quad (116)$$

We apply our Price inequality to prove a vanishing result for L^2 -Betti numbers of manifolds without conjugate points and negative Ricci curvature.

Theorem 117. *Let (M^n, g) be a closed Riemannian manifold without conjugate points and $-1 \leq \sec_g \leq 1$. If there exists $\delta > 4k^2$ such that*

$$-Ric \geq \delta g,$$

then $b_k^{(2)}(M) = 0$.

Proof. The claimed vanishing follows from Corollary 114 combined with (116). \square

Theorem 117 provides new evidence for the Singer Conjecture. Let us recall its statement.

Conjecture 118 (Singer Conjecture). *If M^n is a closed aspherical manifold, then*

$$b_k^{(2)}(M) = 0, \quad \text{if } 2k \neq n.$$

This conjecture is still open, even under the assumption that M^n admits a metric with *strictly negative* sectional curvature. While Theorem 117 does not assume the sectional curvature to be negative, it covers an insufficient range of Betti numbers to settle the conjecture. For more on the Singer Conjecture we refer to [Lüc02, Chapter XI].

Remark 119. It is interesting to observe that Theorem 117 when combined with the Lück approximation theorem [Lüc94] can be used to give an alternative proof of Equation (4) in Theorem 2. More precisely, we need to apply first Theorem 117 to closed manifolds with residually finite fundamental group, and then appeal to Lück's approximation theorem. Nevertheless, this L^2 -cohomology approach to Theorem 2 has the disadvantage of not estimating how fast the ratio in (4) converges to zero. On the other hand, our original Price inequality approach to Theorem 2 provides directly an effective estimate for this convergence.

We conclude with an L^2 -Betti number vanishing result for ϵ -pinched negatively curved closed manifolds. This result extends to the case $\epsilon_{n,k} = 0$ a vanishing theorem for ϵ -pinched negatively curved manifolds given by Donnelly-Xavier in [DX84, Proposition 4.1].

Corollary 120. *Let (M^n, g) be a complete simply connected Riemannian manifold of dimension $n_{\mathbb{R}} \geq 4$. Assume the sectional curvature is ϵ -pinched*

$$-(1 + \epsilon)^2 \leq \sec_g \leq -1,$$

with $\epsilon > 0$. Let k be a positive integer such that

$$\epsilon_{n,k} = (n - 1) - 2k(1 + \epsilon) = 0.$$

Then, there are no L^2 -harmonic k -forms $\mathcal{H}_2^k(M) = 0$.

Proof. This vanishing result is an immediate consequence of the Price inequality given in Theorem 93 combined with the fact that $\gamma_g(M) = \infty$. \square

As before, this result implies a vanishing result for L^2 -Betti numbers of certain negatively curved manifolds.

Proposition 121. *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 4$. Assume the sectional curvature is ϵ -pinched*

$$-(1 + \epsilon)^2 \leq \sec_g \leq -1,$$

with $\epsilon > 0$. Let k be a positive integer such that

$$\epsilon_{n,k} = (n - 1) - 2k(1 + \epsilon) = 0,$$

we have $b_k^{(2)}(M) = 0$.

This vanishing result complements the one proved by Donnelly-Xavier in [DX84] by extending it to the critical equality case. More precisely, they prove a vanishing for L^2 -Betti numbers of any degree k such that $\epsilon_{n,k} > 0$. The vanishing in the hyperbolic case ($\epsilon = 0$) for the critical degree $k = \frac{n-1}{2}$ was treated earlier by Dodziuk in [Dod79]. Alternatively, if one wishes, Corollary 104 can be used to give an alternative proof of Dodziuk's vanishing in the critical degree. Once again, the vanishing in Proposition 121 supports Conjecture 118.

References

- [ABBGNRS12] M. Abert, N. Bergeron, I. Biringer, T. Gelander, N. Nikolov, J. Raimbault, I. Samet, On the growth of L^2 -invariants for sequences of lattices in Lie groups. *arXiv:1210.2961v4* [math.RT].
- [Ang93] N. Anghel, An abstract index theorem on non-compact Riemannian manifolds. *Houston J. Math.* **19** (1993), no.2, 223-237.
- [Ati76] M. F. Atiyah, Elliptic operators, discrete groups and von Neumann algebras. *Colloque “Analyse et Topologie” en l’Honneur de Henri Cartan (Orsay, 1974)*, pp. 43-72, Astérisque, No. 32-32, Soc. Math. France, Paris, 1976.
- [BLLS14] N. Bergeron, P. Linnell, W. Lück, and R. Sauer, On the growth of Betti numbers in p -adic analytic towers, *Groups Geom. Dyn.* **8** (2014), no. 2, 311-329.
- [Bor63] A. Borel, Compact Clifford-Klein forms of symmetric spaces. *Topology* **2** (1963), 111-122.
- [DW78] D. DeGeorge and N. Wallach, Limit formulas for multiplicities in $L^2(G/\Gamma)$, I. *Annals of Math.* **107** (1978), no. 1, 133-150.
- [DW79] D. DeGeorge and N. Wallach, Limit formulas for multiplicities in $L^2(G/\Gamma)$, II. *Annals of Math.* **109** (1979), no. 3, 477-495.
- [Dod79] J. Dodziuk, L^2 Harmonic forms on rotationally symmetric Riemannian manifolds. *Proc. Amer. Math. Soc.* **77** (1979), no.3, 395-400.
- [DX84] H. Donnelly, F. Xavier, On the Differential Form Spectrum of Negatively Curved Riemannian Manifolds. *Amer. J. Math.* **106** (1984), no. 1, 169-185.
- [Lüc94] W. Lück, Approximating L^2 -invariants by their finite-dimensional analogues. *Geom. Funct. Anal.* **4** (1994), no.4, 455-481.
- [Lüc02] W. Lück, L^2 -invariants: theory and applications to geometry and K -theory. Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics, 44. Springer-Verlag, Berlin, 2002.
- [Mar14] S. Marshall, Endoscopy and cohomology growth on $U(3)$. *Compos. Math.* **150** (2014), no.6, 903-910.
- [Pet15] P. Petersen, Riemann Geometry. Fourth Edition. Graduate Texts in Mathematics, 171. Springer-Verlag, New York, 2015.
- [Pri83] P. Price, A monotonicity formula for Yang-Mills fields. *Manuscripta Math.* **43** (1983), no. 2-3, 131-166.

- [SX91] P. Sarnak, X. Xue, Bounds on the multiplicities of automorphic representations. *Duke Math. J.* **64** (1991), no. 1, 207-227.
- [Xue91] X. Xue, On the first Betti numbers of hyperbolic surfaces. *Duke Math. J.* **64** (1991), no.1, 85-110.
- [Yeu94] S.-K. Yeung, Betti numbers on a tower of coverings. *Duke Math. J.* **73** (1994), no.1, 201-225.